m, a, and b may be any integers, subject to the conditions for positive values.

$a=0$, when $m=n$.

The least values are obtained by taking $m=a=1$, and $b=2$, and dividing by 3^2, the highest common square factor.

Whence $x=17$, $y=32$, $z=32$.

These values may also be found by taking $a=0$, and $m=b=1$. Then $x=32$, $y=32$, $z=17$.

The least *different* values are obtained by taking $m=a=1$, and $b=4$, using $b+2m+a$, and dividing by 3^2.

Whence $x=41$, $y=80$, $z=320$.

By using $b-(2m+a)$, in the last case, we find $x=9$, $y=16$, $z=0$.

Excellent solutions of this problem were received from *PROFESSORS ZERR, CROSS*, and *WALKER*, and the late *JOSIAH H. DRUMMOND*. Mr. Cross sent in two solutions, one of which was a solution of the generalized problem. If space permits, his solution of the generalized problem will be published in the next issue of the MONTHLY.

Professor Walker should have been credited with a solution of problem 106.

No solutions of problems 105 and 108 have yet been received.

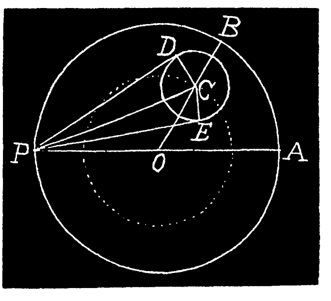
AVERAGE AND PROBABILITY.

90. Proposed by WALTER H. DRANE. Graduate Student, Harvard University.

During a rain-storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond?

III. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the following solution, we assume that the path of the man is along a random straight line drawn from a random point in the circumference of the field. We also assume that the number of favorable paths is to the total number of paths as the arc of a circle (radius, the line drawn from the random point on the circumference of the field to the center of the pond) intercepted by the pond, is to the semi-circumference of the same circle, and that all directions of the path are equally probable; that all values of the radius of the pond less than the radius of the field are also equally probable; that all points on the circumference of the field are equally likely to become the point of starting across the field; and that all points of the field are equally likely to become the center of the pond.



Let O be the center of the field, radius $AO=R$; C, the center of the pond; and P, the point where the man enters the field.

Let $x=OC$, the distance from the center of the field to the center of the pond; $z=CD=CE$, the radius of the pond; $\theta=\angle AOB$; and $\phi=\angle CPE=\angle CPD=\sin^{-1}\left(\dfrac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)$

Then, (1) the chance that the center of the pond lies on the area comprised between two

concentric circles whose common center is O and whose radii are x and $x+dx$ is $C_1=\frac{2\pi x dx}{\pi R^2}=\frac{2xdx}{R^2}$; (2) the chance that the radius of the pond lies between z and $z+dz$ is $C_2=\frac{dz}{R-x}$; (3) the chance that the line of centers, OC, makes an angle with the diameter, AP, between θ and $\theta+d\theta$ is $C_3=d\theta/\pi$; (4) the chance that the path of the man lies between ϕ and $\phi+d\phi$ is $C_4=d\phi/\pi$.

The chance of the concurrence of all of these events is $C_0=C_1C_2C_3C_4$, and the chance of the concurrence of all these events for all values of the variables is

$$C=\int C_1\int C_2\int C_3\int C_4$$

The limits of θ are 0 and π and doubled; the limits of x are 0 and R; the limits of z are 0 and $R-x$; and the limits of ϕ are 0 and ϕ.

$$\therefore\ C=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{dz}{R-x}\int_0^\phi d\phi=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R xdx\int_0^{R-x}\frac{dz}{R-x}\phi$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R xdx\int_0^{R-x}\frac{1}{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)dz$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R\frac{xdx}{R-x}\left[z\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)+\sqrt{R^2+x^2+2Rx\cos\theta}\right]_0^{R-x}$$

$$=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\int_0^R[x\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)+2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta$$

$$-\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}]\,dx=\frac{2}{\pi^2R^2}\int_0^\pi d\theta\left[\tfrac{1}{2}x^2\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)\right]_0^R d\theta$$

$$+\tfrac{1}{2}\int_0^R dx\int_0^\pi\frac{x(R+x)\sqrt{(Rx)}\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}d\theta+2\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}\int_0^\pi\cos\tfrac{1}{2}\theta d\theta$$

$$-\int_0^\pi d\theta\int_0^R\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx=\frac{2}{\pi^2R^2}\left\{\int_0^R x\log\left(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right)dx\right.$$

$$\left.+4\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}-\int_0^\pi d\theta\int_0^R\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx\right\}$$

$$=\frac{2}{\pi^2R^2}\left\{\tfrac{1}{2}x^2\log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right]_{x=R}-\tfrac{1}{2}\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}+4\sqrt{R}\int_0^R\frac{x^{\frac{3}{2}}dx}{R-x}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx \Big\} = \frac{2}{\pi^2 R^2}\Big\{ \tfrac{1}{2}x^2 \log \frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$+\frac{z}{2}\sqrt{R}\Big[-\tfrac{2}{3}x^{\frac{3}{2}} - 2Rx^{\frac{1}{2}} + R^{\frac{3}{2}}\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]\Big]_0^R$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx\Big\}$$

$$=\frac{2}{\pi^2 R^2}\Big\{-\tfrac{28}{3}R^2 + \Big[\tfrac{1}{2}(x^2+7R^2)\log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx\Big\}$$

$$=\frac{2}{\pi^2 R^2}\Big\{-\tfrac{28}{3}R^2 + \Big[\tfrac{1}{4}(x^2+7R^2)\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]_{x=R}\int_0^\pi \cos(\tfrac{1}{2}\theta)\,d\theta$$

$$+\int_0^\pi d\theta\,\Big[\tfrac{1}{2}(x+R\cos\theta)\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$+\tfrac{1}{2}R^2\sin^2\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+x+R\cos\theta]+R\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$-2R^2\cos^2\tfrac{1}{2}\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+R-x-2R\cos^2\tfrac{1}{2}\theta]-2R^2\cos\tfrac{1}{2}\theta\,\times$$

$$\log\Big(\frac{\sqrt{(R^2+x^2+2Rx\cos\theta)}+2R\cos\frac{1}{2}\theta)}{R-x}-\cos\tfrac{1}{2}\theta\Big)\Big]_0^R$$

$$=\frac{2}{\pi^2 R^2}\Big[-\tfrac{28}{3}R^2+R^2\int_0^\pi(2\cos^2\tfrac{1}{2}\theta-\tfrac{1}{2}\cos\theta+2\cos\tfrac{1}{2}\theta-1)\,d\theta$$

$$+R^2\int_0^\pi(\tfrac{1}{2}\sin^2\theta+2\cos^2\tfrac{1}{2}\theta+2\cos\tfrac{1}{2}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta\Big]$$

$$=\frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_0^\pi$$

$$-\int_0^\pi(-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\frac{\sin(\frac{1}{2}\theta)\,d\theta}{2\cos\frac{1}{2}\theta(1+\cos\frac{1}{2}\theta)}\Big\},$$

$$=\frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta=\pi}+\tfrac{1}{4}\int_0^\pi(1-\cos\tfrac{1}{2}\theta)\cos\theta\,d\theta$$

$$-\int_0^\pi(1-\cos\tfrac{1}{2}\theta)\,d\theta-2\int_0^\pi\Big(\frac{1-\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta-\tfrac{5}{4}\int_0^\pi\frac{\theta(1-\cos\frac{1}{2}\theta)\,d\theta}{\sin\theta}\Big\},$$

$$=\frac{2}{\pi^2}\Big[-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta\pm\pi}-\tfrac{1}{6}-(\pi-2)-\Big[4\log\Big(\frac{1+\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)$$

$$-2\theta\Big]_{\theta\pm\pi}-\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{2}\theta\Big]_0^\pi-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{2}\theta)d\theta^*+\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{4}\theta\Big]_0^\pi$$

$$-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{4}\theta)d\theta\Big\}=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log 2-\tfrac{5}{4}\int_0^\pi\log\tan(\tfrac{1}{4}\theta)d\theta\Big)=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log 2$$

$$-5\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta\Big)=\frac{2}{\pi^2}\Big(40\sum_{n=1}^{n=\infty}\frac{2n-1}{(4n-3)^2(4n-1)^2}-\tfrac{5}{6}-4\log 2\Big)$$

$$=\frac{2}{\pi^2}(40\times .114488335\dagger-\tfrac{5}{6}-4\log 2)=\frac{2}{\pi^2}(4.5795334-\tfrac{5}{6}-4\log 2).$$

NOTE.—The above solution of the "pond problem" was attempted at the same time that the published solution of this problem was prepared, but owing to an error in finding an approximate value of

$$\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta$$

the result obtained, appeared to be negative. The solution of problem 156, Calculus Department, involved the evaluation of this same definite integral and in finding the approximate value of this integral, again the result was put in the former answer of the "pond problem" and thus led to a consistent answer.

At the time the former solutions of this problem were published it was my intention to publish this solution, for the reason that about thirty years ago, a controversy arose between Mr. Henry Heaton, of Atlantic, Iowa, and Dr. Joel E. Hendricks, Editor of the *Analyst*, concerning the method of solution of problem 135, of the *Analyst*, this problem involving the same point of difference of solution as is involved in the above solution and my former solution.

In this controversy, Mr. Heaton championed the method followed out in the above solution, and Dr. Hendricks defended the method of the two former published solutions. Mr. Heaton tells us that Professor E. B. Seitz was first with Dr. Hendricks, but was finally convinced by the argument of Mr. Heaton conveyed to him through correspondence. Professor Seitz then converted Dr. Hendricks.

The controversy was again renewed when in 1877, Mr. Heaton sent to Dr. Artemas Martin, now of Washington, D. C., a solution of problem 27, *Mathematical Visitor*, a journal edited and published by Dr. Martin himself,—this problem involving the same bone of contention. Mr. Heaton says, "When I sent Mr. Martin my solution, he wrote back that it was wrong, and said that he had a correct solution from another contributor, and that Professor Benjamin Pierce, and Professor Woolhouse, of England, had decided my solution wrong. I immediately wrote Professor Pierce inclosing Mr. Martin's letter and my solution." In reply to this letter, Mr. Heaton received the following letter from Professor Pierce, the original of which Mr. Heaton kindly loaned me:

MY DEAR SIR:

Mr. Martin informed you correctly concerning my decision. But upon reconsideration, I have decided that I must reverse it, and I now regard your solution as correct. I shall write the grounds of my reversal to Mr. Martin.

Yours faithfully,

BENJAMIN PIERCE.

Thus the controversy ended, and many problems in probability have been solved indifferently by one method or the other from that time to the present.

The solution of the "pond problem" again raises the question as to which method is correct. Certainly, both methods can not be correct, since they lead to different results, when the assumptions as to the distribution of the several events are granted in any particular way.

In the first place, the problem is indefinite, since as many different solutions may be obtained as there are ways of interpreting it and assigning the laws of distribution of the several events. When these have once been assumed then all solutions by whatever method should lead to the same result.

Suppose that the distribution of the events are assumed as in the solution above. Then the solution

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*See problem 123, Calculus Department.

†This is the value computed by Professor Zerr, problem 156, Calculus Department.

of the problem is based upon the definition of the probability of an event which is the *total number of favorable cases divided by the total number of cases.*

In this problem, the favorable cases have been assumed to be proportional to the length of the arc $\phi\times PE$ and the total number of cases proportional to the length of the semi-circumference $\pi\times PE$.

Hence, the chance that the man crosses the pond for any particular value of ϕ, x and θ being constant, is

$$C_1=\frac{\phi\times PE}{\pi\times PE}=\phi/\pi=\frac{\sin^{-1}[z\div(R^2+2x^2+2Rx\cos\theta)]}{\pi}=\frac{f(z)}{\pi}.$$

Now, all values of $f(z)$ between the limits $z=0$ and $z=R-x$ must be considered. Let

$$h=\frac{0+R-x}{n}=\frac{R-x}{n}.\quad \text{Then,}\ \sum_{z=0}^{z=R-x}f(z)=f(h)+f(2h)+f(3h)+\ldots.+f(nh).$$

The chance that $f(z)$ has the value of any particular term as $f(kh)$ of this series is $1/n$, or $h/(R-x)$.

Hence, the chance that the man crosses the pond is the product of the chance that $f(kh)$ is the value of $f(z)$.multiplied by $f(kh)/\pi=f(kh)/[\pi(R-x)]$.

Hence, the chance that the man crosses the pond for all values of $f(z)$ between the limits $z=0$ and $z=R-x$, x and θ being constant, is

$$C_2=\frac{f(h)+f(2h)h+f(3h)h+\ldots+f(nh)h}{\pi(R-x)}=\frac{\int_0^{R-x}f(z)dz}{\pi(R-x)}=F(x).$$

$F(x)$ is the chance of the man crossing the pond when the center of the pond is at a certain fixed point. But the center of the pond may be any point in the field. The chance that the center of the pond falls on any point between two concentric circles whose radii are x and $x+dx$ is $\pi xdx/\pi R^2=2xdx/R^2$.

Hence, the chance of the man crossing the pond is $(2xdx/R^2)F(x)$, and for all values of x between $x=0$ to $x=R$ the chance is

$$C_3=\int_0^R\frac{2xdx}{R^2}F(x)=\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{f(z)dz}{\pi(R-x)}=\Psi(\theta).$$

The chance that θ has any value between θ and $\theta+d\theta$ is $d\theta/\pi$.

Hence, the chance that the man crosses the pond for all values of $\psi(\theta)$ between $\theta=0$ to θ is

$$C_4=\int_0^\pi\frac{d\theta}{\pi}\Psi(\theta)=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}F(x)=\int_0^\pi\frac{d\theta}{\pi}\int_0^R\frac{2xdx}{R^2}\int_0^{R-x}\frac{f(z)dz}{\pi(R-x)}.$$

Thus it appears that the first method when made to obey strictly the definition of probability leads to the same form for computation as the second. The first two solutions of this problem as published in the December Number, of Vol. VII, of the MONTHLY, are, therefore, incorrect.

Dr. Martin informs me that Professor Seitz sent him a solution of this problem and obtained the same result as that given in the two solutions previously published. Professor Seitz, in his day, solved a great many very difficult probability problems, the solutions of very few of which have since been found erronous, if the laws of distribution assumed by him be granted.

The above solution of the "pond problem" has the sanction of the ablest living mathematicians in this country.

This point, viz., that the total number of events are not to be obtained in such problems by writing for the denominator a certain set of definite integrals, should not be overlooked in solving probability problems of this particular in the future.

130. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the the points taken three and three, is 1-35 of the volume of the given sphere.